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Exactly solvable position-dependent mass Hamiltonians related to non-compact semi-simple Lie groups

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Abstract
We suggest a generalized procedure to obtain exactly solvable position-dependent mass Hamiltonians in one dimension. The second-order Casimir invariant of the regular representation of a non-compact semi-simple Lie group G, the spectral properties of which are well known, is used to introduce exactly solvable Hamiltonians. A brief description of the procedure is presented and its application to quantum systems associated with $SL(2, R)$ is detailed.

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1. Introduction

In recent years, quantum-mechanical systems with a position-dependent mass have gained renewed interest owing to the relation to effective-mass approximation in condensed matter physics [1–3]. They are very useful in the study of electronic properties of semiconductors [4], quantum dots [5], compositionally graded crystals [6], etc. The position-dependent mass concept also appears in the energy-density functional approach to quantum many-body systems, such as nuclei [7], quantum liquids [8], metal clusters [9], ^{3}He clusters [10], etc.

These applications have stimulated the search for exactly solvable Hamiltonians with a position-dependent mass. A number of authors have studied the position-dependent mass Schrödinger equation within the framework of super-symmetric quantum mechanics [11–15], the point-canonical transformation approach [16–19], Lie algebraic methods [20–23], etc.

In this paper, we provide a generalized procedure to obtain exactly solvable position-dependent mass Hamiltonians related to semi-simple Lie groups G. The wavefunctions of such Hamiltonians are given in terms of matrix elements of the underlying Lie group. The procedure is illustrated by explicit application to $SL(2, R)$.

2. Main idea

Let us start the discussion with the fact that, for a position-dependent mass $M(x)$, since $M(x)$ and the momentum operator $\hat{P} = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x}$ no longer commute, there are many possibilities for the kinetic Hamiltonian [24–28]. All of them are special cases of the most general von Roos form [29] given by

$$H_{\text{kin}} = \tfrac{1}{4}(M^a \hat{P} \circ M^b \hat{P} \circ M^c + M^c \hat{P} \circ M^b \hat{P} \circ M^a), \tag{1}$$

with $a + b + c = -1$, where $\circ$ denotes composition of operators. Henceforth, we will set $M(x) = m_0 m(x)$, where $m(x)$ is a dimensionless mass, and adopt units such that $\hbar = 2m_0 = 1$.

In a paper [30], Levi-Leblond proposed a way to solve the ambiguity in the definition of H_{kin}. It has been shown that invariance under instantaneous Galilean transformations leads without ambiguity to a BenDaniel–Duke [24] form (corresponding to $a = c = 0$, $b = -1$) for the Hamiltonian

$$\begin{aligned} H &= -\frac{\mathrm{d}}{\mathrm{d}x} \circ \frac{1}{m(x)} \frac{\mathrm{d}}{\mathrm{d}x} + V(x) \\ &= -\frac{1}{m(x)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\dot{m}(x)}{m^2(x)} \frac{\mathrm{d}}{\mathrm{d}x} + V(x), \end{aligned} \tag{2}$$

where dot represents derivative with respect to x, i.e., $\dot{m} = \frac{\mathrm{d}m}{\mathrm{d}x}$. Here we shall work with this form of the Hamiltonian. The corresponding Schrödinger equation is given by

$$\left[-\frac{1}{m(x)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\dot{m}(x)}{m^2(x)} \frac{\mathrm{d}}{\mathrm{d}x} + V(x) \right] \psi(x) = E\psi(x). \tag{3}$$

A key concept in group-theoretic approach is that the Hamiltonian H under consideration is expressed in terms of infinitesimal operators of some Lie group G. We point out that in this study one must have either an irreducible representation of G or a reducible one whose irreducible content is known. From the beginning we must stress that we deal with systems whose Hamiltonians H can be written as

$$H = (\alpha_1 C + \alpha_2)|_{\mathcal{H}}, \tag{4}$$

where C a second-order Casimir operator C of a reducible representation of non-compact group G. Here $|_{\mathcal{H}}$ denotes the restriction to a subspace $\mathcal{H}$ of carrier space of the representation.

The Schrödinger energy eigenvalue equation for such Hamiltonians is essentially the condition imposed on the carrier space of G to be irreducible. However, it has to be remarked that, even though the knowledge of the irreducible content of the representation allows one to obtain the whole spectrum of the Hamiltonian, generally it is not enough to obtain solutions of Schrödinger equation algebraically. Therefore, we find it expedient to use representations of groups G by shift operators on spaces of functions on G itself. The reason for this is that the eigenfunctions of the corresponding Hamiltonians can be given in terms of the matrix elements of G.

Let $\mathcal{L}^\infty(G)$ be the space of infinitely differentiable functions on the group G. A simple verification shows that the formula

$$T(g_0) f(g) = \left[\frac{h(gg_0)}{h(g)} \right]^{\frac{1}{2}} f(gg_0), \qquad g_0 \in G, \tag{5}$$

where

$$h(g) \geqslant 0 \tag{6}$$

gives a representation of G. The representation T on $L^2(G, \mathrm{d}\mu)$ is unitary with respect to the inner product

$$(f, f') = \int f^*(g) f'(g)\, \mathrm{d}\mu,$$

where $\mathrm{d}\mu = h(g)\,\mathrm{d}g$ and $\mathrm{d}g$ is an invariant measure on G. In the case of $h(g) = 1$, the representation operator, call it $\check{T}$, has the simple form

$$\check{T}(g_0)\check{f}(g) = \check{f}(gg_0). \tag{7}$$

It is called the regular representation of G [31, 32]. This representation is decomposed into unitary irreducible representations (UIRs) of the group G. It should be noted that [33], in the case of non-compact group G, the discrete series representations appear in the decomposition if and only if G has the same rank as its maximal compact subgroup.

The representation $\check{T}$, of course, is equivalent to T. The mapping W which realizes the equivalence is given by

$$W : f \to \check{f} = h^{1/2} f. \tag{8}$$

(Although equivalence from the point of view of mathematics is not equivalence from the point of view of physics.) The next step in the program is the calculation of the second-order Casimir operators of T.

Let $e_1, e_2, \ldots, e_n$ form a basis of the Lie algebra $\mathfrak{g}$ of G with commutation relations

$$[e_i, e_j] = c_{ij}^k e_k, \qquad i, j, k = 1, 2, \ldots, n,$$

where c_{ij}^k are the structure constants, and let Ω_k be one-parameter subgroups generated by e_k, $k = 1, 2, \ldots, n$. Then the infinitesimal operators J_k corresponding to e_k are defined by

$$J_k = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau} T(\omega_k(\tau))\bigg|_{\tau=0}, \qquad \omega_k \in \Omega_k. \tag{9}$$

They satisfy the commutation relations

$$[J_i, J_j] = c_{ij}^k J_k. \tag{10}$$

Moreover, a second-order Casimir operator C of T is given by

$$C = g^{ik} J_i J_k \tag{11}$$

where

$$g_{ik} = c_{ij}^l c_{kl}^j$$

Thus, in order to find the second-order Casimir operator of T, it is sufficient to know infinitesimal operators J_k.

Before calculating infinitesimal operators, we need to know a parametrization of arbitrary elements of G. Many parametrizations of the general element g of G are possible. The most useful ones would appear to arise from factorization of group elements into products of three factors

$$g = bad, \tag{12}$$

each factor constituting a subgroup of G. (It should be noted that different factorizations of G lead to different reductions of T.) We shall assume that the general element g depends on position coordinates of particles only through the middle factor a. Once the parametrization of arbitrary elements is given, it is almost straightforward to get an explicit form of the infinitesimal operators. Next one can extract a family of position-dependent mass Hamiltonians from the Casimir operator as follows.

Let $\mathcal{H}$ be a subspace of functions $f(g)$ such that

$$f(bgd) = u_r(b)w_s(d)f(g), \tag{13}$$

with fixed r and s, where u_r and w_s are the UIR matrix elements of subgroups consisting of the matrices b and d, respectively. Then any function f of this subspace is of the form

$$f(g) = u_r(b)w_s(d)f(a), \quad a \in A. \tag{14}$$

We now require $f(a)$ to be square-integrable functions. Moreover we shall assume that h in (5) depends on position coordinates alone. Then the Casimir operator restricted to $\mathcal{H}$ becomes a differential operator in position coordinates alone, yielding a family of position-dependent mass Hamiltonians by appropriate choice of h.

3. A case study for $SL(2, R)$

To gain a better understanding of our approach, we illustrate it for Hamiltonians related to $SL(2, R)$. In order to fix notation and terminology we start with a brief description of UIRs of the group $SL(2, R)$. For more detailed treatment of $SL(2, R)$ we refer to [32].

$SL(2, R)$ is the group of all 2×2, real, unimodular matrices. Its Lie algebra $sl(2, R)$ consists of traceless real 2×2 matrices. We choose the basis in $sl(2, R)$ as

$$e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{15}$$

The commutation relations for these matrices are

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = -e_1, \qquad [e_3, e_1] = -e_2. \tag{16}$$

According to this, we choose three one-parameter subgroups Ω_1, Ω_2 and Ω_3 generated by e_1, e_2 and e_3, respectively. They consist of the matrices of the form

$$\omega_1 = \begin{pmatrix} \mathrm{e}^{-\tau/2} & 0 \\ 0 & \mathrm{e}^{\tau/2} \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix}, \qquad \omega_3 = \begin{pmatrix} \cos\frac{\tau}{2} & -\sin\frac{\tau}{2} \\ \sin\frac{\tau}{2} & \cos\frac{\tau}{2} \end{pmatrix}, \tag{17}$$

respectively.

The unitary irreducible representations of $SL(2, R)$ are known to form three series: principal, supplementary and discrete. (It should be noted that rank $SL(2, R) =$ rank $SO(2) = 1$.) The Casimir operator

$$C = J_1^2 + J_2^2 - J_3^2 \tag{18}$$

for all such UIRs is identically a multiple of the unit

$$C = -j(j+1)I,$$

where J_1, J_2 and J_3 are the Hermitian operators corresponding to e_1, e_2, e_3, respectively. J_3 is elliptic, J_1, J_2 are hyperbolic. The representations specified by j and $-1-j$ are equivalent.

When we use a $SO(2)$ basis, J_3 will be the preferred generator. We now give the spectrum of j corresponding to UIRs and eigenvalues n of the operator J_3 in each such representation:

(i) principal series $T_{\mathrm{i}\rho-\frac{1}{2}}$

$$j = -\tfrac{1}{2} + \mathrm{i}\rho, \quad 0 \leqslant \rho < \infty, \quad m = 0, \pm 1, \pm 2, \ldots, \quad \text{or} \quad m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots, \tag{19}$$

(ii) complementary series T_τ

$$j = \tau, \quad -1 < \tau < -\tfrac{1}{2}, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{20}$$

(iii) positive discrete series T_j^+

$$j = -\tfrac{1}{2}, -1, -\tfrac{3}{2}, -2, \ldots, \qquad n = -j, -j+1, -j+2, \ldots, \tag{21}$$

(iv) negative discrete series T_j^-

$$j = -\tfrac{1}{2}, -1, -\tfrac{3}{2}, -2, \ldots, \qquad n = j,\ j-1,\ j-2, \ldots. \tag{22}$$

Two other subgroup bases are $SO(1,1)$ and $E(1)$ bases in which, for example, the operators J_1 and $N_+ = (J_2 - J_3)/2$ are diagonal, respectively. It is worth pointing out that eigenvalues ν and λ of the operators J_1 and N_+ are real numbers in each UIR.

We are now prepared to obtain the explicit form of Hamiltonians related to $SL(2,R)$ in the sense that relation (4) holds.

(i) *Cartan decomposition*

Here we consider a parametrization of $SL(2,R)$ group, which is consistent with Cartan decomposition [32]

$$g(\varphi,t,\theta) = \begin{pmatrix} \cos\varphi/2 & \sin\varphi/2 \\ -\sin\varphi/2 & \cos\varphi/2 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}.$$

One can represent almost every element $g \in SL(2,R)$ in this form by imposing upon φ, t, θ the restrictions

$$0 \leqslant \varphi < 2\pi, \qquad 0 \leqslant t < \infty, \qquad -2\pi \leqslant \theta < 2\pi.$$

Moreover, the correspondence $(\varphi,t,\theta) \to g(\varphi,t,\theta)$ is one-to-one. Now we require the parameter t to be a differentiable function of position x. Moreover we shall assume that h in (5) depends on x alone. Then it follows from equations (9) that

$$\begin{aligned} \mathrm{i}J_1 &= -\frac{\cos\theta}{\dot t}\frac{\partial}{\partial x} - \frac{\sin\theta}{\sinh t}\frac{\partial}{\partial\varphi} + \sin\theta\coth t\frac{\partial}{\partial\theta} - \frac{\dot h}{2h\dot t}\cos\theta, \\ \mathrm{i}J_2 &= \frac{\sin\theta}{\dot t}\frac{\partial}{\partial x} - \frac{\cos\theta}{\sinh t}\frac{\partial}{\partial\varphi} + \cos\theta\coth t\frac{\partial}{\partial\theta} + \frac{\dot h}{2h\dot t}\sin\theta, \\ \mathrm{i}J_3 &= -\frac{\partial}{\partial\theta}. \end{aligned} \tag{23}$$

An invariant measure $\mathrm{d}g$ in this parametrization is given by $\mathrm{d}g = \dot t \sinh t\, \mathrm{d}x\, \mathrm{d}\varphi\, \mathrm{d}\theta$. If we compute the Casimir operator C for this parametrization, it becomes

$$\begin{aligned} C = &-\frac{1}{\dot t^2}\frac{\partial^2}{\partial x^2} - \frac{1}{\dot t^2}\left(\frac{\dot h}{h} - \frac{\ddot t}{\dot t} + \dot t\coth t\right)\frac{\partial}{\partial x} - \frac{1}{\sinh^2 t}\left(\frac{\partial^2}{\partial\varphi^2} - 2\cosh t\frac{\partial^2}{\partial\varphi\partial\theta} + \frac{\partial^2}{\partial\theta^2}\right) \\ &+ \frac{\dot h}{2h\dot t^2}\left(\frac{\dot h}{2h} - \frac{\ddot h}{\dot h} + \frac{\ddot t}{\dot t} - \dot t\coth t\right). \end{aligned} \tag{24}$$

Let $\mathcal{H}_{nk}$ be a subspace of $L^2(SL(2,R), \mathrm{d}\mu)$ consisting of functions $f(g)$ such that

$$f(g) = e^{-\mathrm{i}(n\varphi + k\theta)} f(t), \tag{25}$$

where n and k are simultaneously integer or half-integer. (We note that $e^{-\mathrm{i}n\varphi}$ ($e^{-\mathrm{i}k\theta}$) is a UIR of the $SO(2)$ subgroup.) Then the Casimir operator restricted to $\mathcal{H}_{nk}$, call it C_{nk}, becomes a differential operator in x alone

$$\begin{aligned} C_{nk} = &-\frac{1}{\dot t^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{\dot t^2}\left(\frac{\dot h}{h} - \frac{\ddot t}{\dot t} + \dot t\coth t\right)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{n^2 + k^2 - 2nk\cosh t}{\sinh^2 t} \\ &+ \frac{\dot h}{2h\dot t^2}\left(\frac{\dot h}{2h} - \frac{\ddot h}{\dot h} + \frac{\ddot t}{\dot t} - \dot t\coth t\right). \end{aligned} \tag{26}$$

As we want to get a Hamiltonian in the form (2), we require

$$h = \frac{1}{\dot{t}\sinh t}. \tag{27}$$

Then it can be shown that the position-dependent mass Hamiltonian (2) with solvable potentials

$$V(x) = \frac{(n-k)^2 - \frac{1}{4}}{\sinh^2\frac{t}{2}} - \frac{(n+k)^2 - \frac{1}{4}}{\cosh^2\frac{t}{2}} + \frac{\dddot{t}}{2\dot{t}^3} - \frac{5}{4}\frac{\ddot{t}^2}{\dot{t}^4} \tag{28}$$

is related to Casimir invariant C as

$$H = \left(C - \tfrac{1}{4}\right)\big|_{\mathcal{H}_{nk}} \tag{29}$$

with

$$m(x) = \dot{t}^2(x). \tag{30}$$

Moreover, it is well known that the functions $\check{f}(t)$ such that

$$\int_0^\infty |\check{f}(t)|^2 \sinh t \,\mathrm{d}t < \infty \tag{31}$$

have the expansion [32]

$$\check{f}(t) = \int_0^\infty c(\rho) t_{nk}^{-\frac{1}{2}+\mathrm{i}\rho}(g(0,t,0)) \tanh\pi(\rho + \mathrm{i}\epsilon)\,\mathrm{d}\rho + \sum_{j=-1-\epsilon}^{-N} \left(-j - \frac{1}{2}\right) c_j t_{nk}^{j,\pm}(g(0,t,0))$$

with Fourier coefficients $c(\rho)$ and c_j, where $N = \min(|n|, |k|)$ for $nk > 0$ (the sum over j is absent for $nk \leqslant 0$), $\epsilon = 0$, if n and k are integer, and $\epsilon = \frac{1}{2}$, if n and k are half-integer. Here $t_{nk}^{-\frac{1}{2}+\mathrm{i}\rho}(g)$ and $t_{nk}^{j,\pm}(g)$ are matrix elements of continuous and discrete series representations of $SL(2,R)$ in $SO(2)$ basis, respectively. Hence, the potential (28) has scattering states for any (simultaneously integer or half-integer) n, k, whereas it possesses bound states if $nk > 0$.

The energy spectrum can now be obtained easily if we note that the eigenvalue of C is $-j(j+1)$. Thus we have

$$E = -\left(j + \tfrac{1}{2}\right)^2 \tag{32}$$

where $j = -\frac{1}{2} + \mathrm{i}\rho$ for the scattering states, whereas

$$j = -\frac{3}{2}, -\frac{5}{2}, \ldots, -N \quad \text{(if } k \text{ and } n \text{ are half-integer and } nk > 0) \tag{33}$$

or

$$j = -1, -2, \ldots, -N \text{ (if } k \text{ and } n \text{ are integer and } nk > 0) \tag{34}$$

for the bound states. We may also use the UIR matrix elements of $SL(2,R)$ in $SO(2)$ basis [31] to define the wavefunctions of (28). In particular, for bound state wavefunctions we have

$$\psi(x) \propto (\dot{t}\sinh t)^{1/2} \left(\cosh\frac{t}{2}\right)^{2j} \left(\tanh\frac{t}{2}\right)^{k-n} \times F\left(-j+k, -j-n; k-n+1; \tanh^2\frac{t}{2}\right) \tag{35}$$

where $n \leqslant k \leqslant j < -\frac{1}{2}$. (If $k \leqslant n \leqslant j < -\frac{1}{2}$ one has to replace n by k and k by n.) It should be noted that due to the symmetry $n \to -n, k \to -k$ in the potentials (28), both the positive and negative series describe the same bound states.

What is still missing though is a group-theoretical derivation of the scattering matrix. It can be shown that S-matrix of such quantum-mechanical systems can also be derived via intertwining operators of underlying Lie group [34]. This subject will be treated in a later publication.

The mass and potential functions depend crucially on $t(x)$. So to obtain specific mass and potential functions it is necessary to make a choice of the function $t(x)$. For example, to obtain the mass function

$$m(x) = \left(1 + \frac{\delta}{1 + x^2}\right)^2 \tag{36}$$

used in [12] we have to set for $t(x)$ the following form:

$$t(x) = x + \delta \arctan x, \qquad 0 \leqslant x < \infty, \quad \delta > 0. \tag{37}$$

With this choice of $t(x)$, potential functions are found to be

$$V(x) = \frac{(n-k)^2 - \frac{1}{4}}{\sinh^2 \frac{t}{2}} - \frac{(n+k)^2 - \frac{1}{4}}{\cosh^2 \frac{t}{2}} + \frac{\delta[3x^4 + 2(1-\delta)x^2 - 1 - \delta]}{(1 + \delta + x^2)^4}. \tag{38}$$

From the discussion of the preceding section, it should be clear that other choices of function $t(x)$ can be made which satisfy the condition

$$\dot{t} > 0, \qquad 0 \leqslant t < \infty$$

(see equations (6) and (27)). In other words, $t(x)$ is a strictly increasing function and its range is $[0, \infty)$. We can take for instance

$$t(x) = \ln\left(\frac{e^x + \delta}{1 + \delta}\right), \qquad 0 \leqslant x < \infty, \quad \delta > 0, \tag{39}$$

which gives

$$m(x) = 1/(1 + \delta\, e^{-x})^2$$

Another acceptable form for $t(x)$ is

$$t(x) = \frac{1}{\sqrt{\delta}} \ln\left(\sqrt{\delta}x + \sqrt{1 + \delta x^2}\right), \qquad 0 \leqslant x < \infty, \quad \delta > 0, \tag{40}$$

with [30]

$$m(x) = 1/(1 + \delta x^2).$$

It can also be seen that

$$\lim_{\delta \to 0} t(x) = x \qquad \text{and} \qquad \lim_{\delta \to 0} m(x) = 1$$

and as a consequence, the hyperbolic Pőschl–Teller potential appears as the $\delta \to 0$ limit of these potentials.

It is also worth pointing out that the trigonometric form of potential (28) can be obtained choosing G as $SU(2)$. In this case, the Cartan decomposition is

$$g(\varphi, t, \theta) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos t/2 & i \sin t/2 \\ i \sin t/2 & \cos t/2 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \qquad g \in SU(2)$$

where $0 \leqslant \varphi < 2\pi, 0 \leqslant t < \pi, -2\pi \leqslant \theta < 2\pi$. Now we choose three one-parameter subgroups $\Omega_1, \Omega_2, \Omega_3$ in $SU(2)$, consisting of the matrices

$$\omega_1 = \begin{pmatrix} \cos\tau/2 & \mathrm{i}\sin\tau/2 \\ \mathrm{i}\sin\tau/2 & \cos\tau/2 \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} \cos\tau/2 & -\sin\tau/2 \\ \sin\tau/2 & \cos\tau/2 \end{pmatrix}, \qquad \omega_3 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\tau/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\tau/2} \end{pmatrix},$$

respectively. Then, it is almost straightforward to get an explicit form of the Casimir operator $C = J_1^2 + J_2^2 + J_3^2$. Next by arguments very similar to those used in arriving at (28) we can show that the Hamiltonian (2) with mass $m = \dot{t}^2(x)$ and potential

$$V(x) = \frac{(n-k)^2 - \frac{1}{4}}{\sin^2\frac{t}{2}} + \frac{(n+k)^2 - \frac{1}{4}}{\cos^2\frac{t}{2}} + \frac{\dddot{t}}{2\dot{t}^3} - \frac{5}{4}\frac{\ddot{t}^2}{\dot{t}^4} \tag{41}$$

is related to $SU(2)$ in the sense that relation

$$H = \left(C + \tfrac{1}{4}\right)\Big|_{\mathcal{H}_{nk}} \tag{42}$$

holds. It should be noted that the function $t(x)$ is strictly increasing on the interval $(0, x_\delta)$ and its range is $(0, \pi)$, where x_δ is the solution of the equation $t(x_\delta) = \pi$. For instance, $x_\delta = \ln[\mathrm{e}^\pi + \delta(\mathrm{e}^\pi - 1)]$ if $m(x) = 1/(1 + \delta\,\mathrm{e}^{-x})^2$.

Moreover, it is well known that the functions $\check{f}(t)$ such that

$$\int_0^\pi |\check{f}(t)|^2 \sin t \,\mathrm{d}t < \infty \tag{43}$$

have the expansion [32]

$$\check{f}(t) = \sum_{l=K}^{\infty} \alpha_l t^l_{nk}(g(0, t, 0))$$

with Fourier coefficients c_l, where $K = \max(|n|, |k|)$ and $t^l_{nk}(g)$ are matrix elements of UIRs of $SU(2)$. (It should be noted that the Casimir operator for UIRs of $SU(2)$ is identically a multiple of the unit $C = l(l+1)$) Hence, the potential (41) has bound states only, with energy

$$E = \left(l + \frac{1}{2}\right)^2, \qquad l = K, K+1, K+2, \ldots.$$

Moreover, the UIRs matrix elements of $SU(2)$ provide the wavefunctions of (41)

$$\psi(x) \propto (\dot{t}\sin t)^{1/2}\left(\sin\frac{t}{2}\right)^{k-n}\left(\cos\frac{t}{2}\right)^{k+n} \times F\left(l+k+1, -l+k; k-n+1; \sin^2\frac{t}{2}\right), \tag{44}$$

where $n \leqslant k$. (If $k \leqslant n$ one has to replace n and k by $-n$ and $-k$, respectively.)

(ii) *The Iwasawa decomposition.* Now, we want to use the Iwasawa decomposition

$$g(\varphi, t, u) = \begin{pmatrix} \cos\varphi/2 & \sin\varphi/2 \\ -\sin\varphi/2 & \cos\varphi/2 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{t/2} & 0 \\ 0 & \mathrm{e}^{-t/2} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \tag{45}$$

where $0 \leqslant \varphi < 2\pi$, $-\infty < t < \infty, -\infty < u < \infty$. We demand t to be a differentiable function of position x. Then

$$\begin{aligned}
\mathrm{i}J_1 &= -\frac{1}{\dot{t}}\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - \frac{\dot{h}}{2h\dot{t}}, \\
\mathrm{i}J_2 &= \frac{u}{\dot{t}}\frac{\partial}{\partial x} - \mathrm{e}^{-t}\frac{\partial}{\partial\varphi} + (1 - u^2 + \mathrm{e}^{-2t})\frac{\partial}{\partial u} + \frac{\dot{h}}{2h\dot{t}}u, \\
\mathrm{i}J_3 &= \frac{u}{\dot{t}}\frac{\partial}{\partial x} - \mathrm{e}^{-t}\frac{\partial}{\partial\varphi} - (1 + u^2 - \mathrm{e}^{-2t})\frac{\partial}{\partial u} + \frac{\dot{h}}{2h\dot{t}}u
\end{aligned} \tag{46}$$

and $\mathrm{d}g = \mathrm{e}^t \dot{t}\, \mathrm{d}x\, \mathrm{d}u\, \mathrm{d}\varphi$. If we compute the Casimir operator C, it becomes

$$C = -\frac{1}{\dot{t}^2}\frac{\partial^2}{\partial x^2} - \frac{1}{\dot{t}^2}\left(\frac{\dot{h}}{h} - \frac{\ddot{t}}{\dot{t}} + \dot{t}\right)\frac{\partial}{\partial x} + 2\mathrm{e}^{-t}\frac{\partial^2}{\partial u \partial \varphi} - \mathrm{e}^{-2t}\frac{\partial^2}{\partial u^2} + \frac{\dot{h}}{2h\dot{t}^2}\left(\frac{\dot{h}}{2h} - \frac{\ddot{h}}{\dot{h}} + \frac{\ddot{t}}{\dot{t}} - \dot{t}\right). \tag{47}$$

Let us denote by $C_{n\lambda}$ a restriction of C on a subspace $\mathcal{H}_{n\lambda}$ of $\mathcal{L}^\infty(G)$ consisting of functions $f(g)$ such that

$$f(g) = \mathrm{e}^{-\mathrm{i}(n\varphi + \lambda u)} f(t),$$

where $\mathrm{e}^{-\mathrm{i}n\varphi}$ and $\mathrm{e}^{-\mathrm{i}\lambda u}$ are UIRs of $SO(2)$ and $E(1)$, respectively. Moreover, we require that the functions $f(t)$ are square-integrable with respect to $h(t)\,\mathrm{e}^t \mathrm{d}t$, i.e.,

$$\int_0^\infty |f(t)|^2 h(t)\,\mathrm{e}^t \mathrm{d}t < \infty. \tag{48}$$

Then, it turns out that

$$C_{n\lambda} = -\frac{1}{\dot{t}^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{\dot{t}^2}\left(\frac{\dot{h}}{h} - \frac{\ddot{t}}{\dot{t}} + \dot{t}\right)\frac{\mathrm{d}}{\mathrm{d}x} - 2\lambda n\,\mathrm{e}^{-t} + \lambda^2\mathrm{e}^{-2t} + \frac{\dot{h}}{2h\dot{t}^2}\left(\frac{\dot{h}}{2h} - \frac{\ddot{h}}{\dot{h}} + \frac{\ddot{t}}{\dot{t}} - \dot{t}\right). \tag{49}$$

Now, we require

$$h = \frac{1}{\dot{t}\mathrm{e}^t}. \tag{50}$$

Then, it is straightforward to show that the Hamiltonian (2) with mass $m(x) = \dot{t}^2(x)$ and potential

$$V(x) = \lambda^2\mathrm{e}^{-2t} - 2\lambda n\,\mathrm{e}^{-t} + \frac{\dddot{t}}{2\dot{t}^3} - \frac{5}{4}\frac{\ddot{t}^2}{\dot{t}^4} \tag{51}$$

is related to the Casimir operator (49) as

$$H = \left(C - \tfrac{1}{4}\right)\big|_{\mathcal{H}_{n\lambda}}. \tag{52}$$

Hence, the energy is also given by (32), but now $j = -\frac{3}{2}, -\frac{5}{2}, \ldots, -|n|$ (if n is half-integer) or $j = -1, -2, \ldots, -|n|$ (if n is integer) for the bound states. Moreover, the $SO(2) \leftrightarrow E(1)$ mixed basis matrix elements [32, 35] provide the bound and scattering state wavefunctions of (51). For instance, the bound states wavefunctions are given by

$$\psi(x) \propto \left(\dot{t}\mathrm{e}^t\right)^{1/2} \exp\left(-\frac{\lambda\mathrm{e}^{-t}}{2}\right) L_{n+j}^{-2j-1}(\lambda^2\mathrm{e}^{-t}), \tag{53}$$

with $n > -j$, where L_n^k are the Laguerre polynomials. Finally, we note that since the range of the function $t(x)$ in (2) is $(-\infty, +\infty)$ the choice of $t(x)$ as in (39) is not acceptable. But we can take $t(x)$ as in (37) or as in (40), where x runs through the domain $(-\infty, +\infty)$.

(iii) *Generalized Cartan decomposition.* Let us adopt the following parametrization for $SL(2, R)$:

$$g(u, t, \varphi) = \begin{pmatrix} \mathrm{e}^{u/2} & 0 \\ 0 & \mathrm{e}^{-u/2} \end{pmatrix} \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix} \begin{pmatrix} \cos\varphi/2 & \sin\varphi/2 \\ -\sin\varphi/2 & \cos\varphi/2 \end{pmatrix},$$

where $-\infty < u < \infty$, $-\infty < t < \infty$, $0 \leqslant \varphi < 2\pi$. As usual we require t to be a differentiable function of x and as a consequence we have

$$\begin{aligned} \mathrm{i}J_1 &= \frac{\sin\varphi}{\dot t}\frac{\partial}{\partial x} - \frac{\cos\varphi}{\cosh t}\frac{\partial}{\partial u} + \tanh t\cos\varphi\frac{\partial}{\partial\varphi} + \frac{\dot h}{2h\dot t}\sin\varphi, \\ \mathrm{i}J_2 &= \frac{\cos\varphi}{\dot t}\frac{\partial}{\partial x} + \frac{\sin\varphi}{\cosh t}\frac{\partial}{\partial u} - \tanh t\sin\varphi\frac{\partial}{\partial\varphi} + \frac{\dot h}{2h\dot t}\cos\varphi, \\ \mathrm{i}J_3 &= -\frac{\partial}{\partial\varphi}, \end{aligned} \tag{54}$$

while $\mathrm{d}g = \dot t\cosh t\,\mathrm{d}x\,\mathrm{d}u\mathrm{d}\varphi$. Then the restriction of the Casimir operator

$$\begin{aligned} C = -\frac{1}{\dot t^2}\frac{\partial^2}{\partial x^2} - \frac{1}{\dot t^2}\left(\frac{\dot h}{h} - \frac{\ddot t}{\dot t} + \dot t\tanh t\right)\frac{\partial}{\partial x} - \frac{1}{\cosh^2 t}\left(\frac{\partial^2}{\partial u^2} - 2\sinh t\frac{\partial^2}{\partial u\partial\varphi} - \frac{\partial^2}{\partial\varphi^2}\right) \\ + \frac{\dot h}{2h\dot t^2}\left(\frac{\dot h}{2h} - \frac{\ddot h}{\dot h} + \frac{\ddot t}{\dot t} - \dot t\tanh t\right) \end{aligned} \tag{55}$$

to a subspace $\mathcal{H}_{\nu n}$ of $\mathcal{L}^\infty(SL(2,R))$ consisting of functions $f(g)$ such that

$$f(g) = \mathrm{e}^{-\mathrm{i}(\nu u + n\varphi)} f(t) \tag{56}$$

where $\mathrm{e}^{-\mathrm{i}\nu u}$ and $\mathrm{e}^{-\mathrm{i}n\varphi}$ are UIRs of $SO(1,1)$ and $SO(2)$, respectively, yields the differential operator $C_{\nu n}$

$$\begin{aligned} C_{\nu n} = -\frac{1}{\dot t^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{\dot t^2}\left(\frac{\dot h}{h} - \frac{\ddot t}{\dot t} + \dot t\tanh t\right)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\nu^2 - n^2 - 2\nu n\sinh t}{\cosh^2 t} \\ + \frac{\dot h}{2h\dot t^2}\left(\frac{\dot h}{2h} - \frac{\ddot h}{\dot h} + \frac{\ddot t}{\dot t} - \dot t\tanh t\right). \end{aligned} \tag{57}$$

In order to put the operator $C_{\nu n}$ in the form (2), we choose

$$h = \frac{1}{\dot t\cosh t} \tag{58}$$

and as a consequence, we come to the Hamiltonian (2) with potential

$$V(x) = (\nu^2 - n^2 + 1/4)\,\mathrm{sech}^2 t - 2\nu n\,\mathrm{sech}^2 t\tanh t + \frac{\dddot t}{2\dot t^3} - \frac{5}{4}\frac{\ddot t^2}{\dot t^4} \tag{59}$$

related to C (55) as in (29). Therefore the energy is also given by (32), with $j = -\frac{3}{2}, -\frac{5}{2}, \ldots, -|n|$ (if n is half-integer) or $j = -1, -2, \ldots, -|n|$ (if n is integer) for the bound states. Moreover, the wavefunctions are provided by the $SO(1,1) \longleftrightarrow SO(2)$ mixed basis matrix elements [32, 35]. In particular, the bound-state wavefunctions are given by

$$\begin{aligned} \psi(x) \propto (\dot t\cosh t)^{1/2}(\cosh t)^{-n}(1 - \mathrm{i}\sinh t)^{\mathrm{i}\nu} \\ \times F\left(-n - j, 1 + j - n; 1 - n + \mathrm{i}\nu; \tfrac{1}{2}(1 - \mathrm{i}\sinh t)\right) \end{aligned} \tag{60}$$

with $n > -j$.

Finally, we note that three other subgroup factorizations yield Hamiltonians possessing a purely continuous spectrum. The scattering states of these Hamiltonians can be related to the principal series representations of $SL(2,R)$ in non-compact bases. For instance, the Gauss decomposition

$$g = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{t/2} & 0 \\ 0 & \mathrm{e}^{-t/2} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \tag{61}$$

where $-\infty < v < \infty$, $-\infty < t < \infty$, $-\infty < u < \infty$, gives

$$V = \mu\lambda\,\mathrm{e}^{-t} + \frac{\dddot{t}}{2\dot{t}^3} - \frac{5}{4}\frac{\ddot{t}^2}{\dot{t}^4}. \tag{62}$$

The scattering state wavefunctions for (62) are provided by the matrix element of principal series representations of $SL(2,R)$ in $E(1)$ basis [32, 35].

4. Conclusion

We presented in this paper a technique to obtain exactly solvable position-dependent mass Hamiltonians within the framework of group theory. Our point of departure is the regular representation of semi-simple Lie groups G. The key idea which allows us to solve the problem is to relate a second-order Casimir invariant of the regular representation with position-dependent mass Hamiltonians. Then the knowledge of harmonic analysis on semi-simple Lie groups G can be used to determine the spectrum and wavefunctions of related Hamiltonians with a position-dependent mass. In particular, the procedure has been applied to obtain solutions of one-body Hamiltonians related to $SL(2,R)$ and $SU(2)$. It is also clear from the procedure that exactly solvable many-body Hamiltonians related to higher-rank Lie groups can also be constructed. We hope to discuss some of these applications in a future publication.

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References

[1] Luttinger J M and Kohn W 1955 *Phys. Rev.* **97** 869
[2] Wannier G H 1957 *Phys. Rev.* **52** 191
[3] Slater J C 1949 *Phys. Rev.* **76** 1592
[4] Bastard G 1992 *Wave Mechanics Applied to Semiconductor Heterostructures* (Les Ulis: Les Editions de Physique)
[5] Harrison P 2000 *Quantum Wells, Wires and Dots* (New York: Wiley)
Serra L I and Lipparini E 1997 *Europhys. Lett.* **40** 667
[6] Geller M R and Kohn W 1993 *Phys. Rev. Lett.* **70** 3103
[7] Ring P and Schuck P 1980 *The Nuclear Many Body Problem* (New York: Springer)
[8] de Saavedra F A, Boronat J, Polls A and Fabriconi A 1994 *Phys. Rev.* B **50** 4248
[9] Puente A A, Serra L I and Casas M 1994 *Z. Phys.* D **31** 283
[10] Baranco M, Pi M, Catica S M, Hernandez E S and Navarro J 1997 *Phys. Rev.* B **56** 8997
[11] Milonovic V V and Ikonic Z 1999 *J. Phys. A: Math. Gen.* **32** 7001
[12] Plastino A R, Rigo A, Casas M, Gracias F and Plastino A 1999 *Phys. Rev.* A **60** 4398
[13] Gönül B, Gönül B, Tutcu D and Özer O 2002 *Mod. Phys. Lett.* A **17** 2057
[14] Quesne C and Tkachuk V M 2004 *J. Phys. A: Math. Gen.* A **37** 4267
[15] Cariñene J F, Rañada M F and Santander M 2004 *Rep. Math. Phys.* **54** 285
[16] Alhaidary A D 2002 *Phys. Rev.* A **66** 042116
[17] Gönül B, Özer O, Gönül B and Üzgün F 2002 *Mod. Phys. Lett.* A **17** 2453
[18] Yu J and Dong S H 2004 *Phys. Lett.* A **325** 194
[19] Bagchi B, Gorain P, Quesne C and Roychoudhury R 2005 *Europhys. Lett.* **72** 155
[20] Roy B and Roy P 2002 *J. Phys. A: Math. Gen.* **35** 3961
[21] Koç R, Koca M and Körcük E 2002 *J. Phys. A: Math. Gen.* **35** 2527
[22] Koç R and Koca M 2003 *J. Phys. A: Math. Gen* **36** 8105
Roy B 2005 *Europhys. Lett.* **72** 1
[23] Bagchi B, Gorain P, Quesne C and Roychoudhury R 2004 *Czech J. Phys.* **54** 1019
[24] BenDaniel D J and Duke C B 1966 *Phys. Rev.* B **152** 683

[25] Gora T and Williams F 1969 *Phys. Rev.* **177** 1179
[26] Bastard G 1981 *Phys. Rev.* B **24** 2593
[27] Zhu Q G and Kroemer H 1983 *Phys. Rev.* B **27** 3519
[28] Li T and Kuhn K J 1993 *Phys. Rev.* B **47** 12760
[29] von Roos O 1983 *Phys. Rev.* B **27** 7547
[30] Levi-Leblond J M 1995 *Phys. Rev.* A **52** 1845
[31] Barut A O and Raczka R *Theory of Group Representations and Applications* (Signapore: World Scientific)
[32] Vilenkin N Ja and Klimyk A U 1991 *Representation of Lie Groups and Special Functions* vol 1 (Dordrecht: Kluwer)
[33] Harish-Chandra 1966 *Acta Math.* **116** 1
[34] Kerimov G A 1998 *Phys. Rev. Lett.* **80** 2976
[35] Basu D and Wolf K B 1982 *J. Math. Phys.* **23** 289